Outline

- confinement in axisymmetry
- quick reminder of Boozer coords.
- quasisymmetry (QS)
  - → definition
  - → classes of solutions
  - → GC Lagrangian in Boozer coordinates
- omnigeneity (QO)
  - → definition
  - → bounce-averaged GC motion
  - → parallel adiabatic invariant
  - → implications
- Comparison of hidden symmetries

Confinement in axisymmetry

- axisymmetry: physical quantities indpt. of ϕ in cylindrical (R, ϕ, z) coords.

Lagrangian for charged-particle motion:

$$L(\vec{r}, \dot{\vec{r}}) = \frac{m\dot{\vec{r}}^2}{2} + q\vec{A}\cdot\dot{\vec{r}}$$

- Write L in cylindrical coords.,

$$\dot{\vec{r}} = \hat{R}\dot{R} + \hat{\phi}R\dot{\phi} + \hat{z}\dot{z},$$

$$\vec{A} = A_R\hat{R} + A_\phi\hat{\phi}R \rightarrow \partial A_R/\partial\phi = \partial A_\phi/\partial\phi = 0$$

- axisymmetry $\rightarrow$ $\partial L/\partial\phi = 0$

E-L: $$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\phi}}\right] = \cancel{\frac{\partial L}{\partial\phi}}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mR^2\dot{\phi} + qA_\phi R = \text{const.}$$

- under strong magnetization limit,

$B \sim \nabla \times A \rightarrow B \sim A/L$

$m R^2 \dot{\phi} \sim m R v_t \qquad q A_\phi R \sim q B R^2$

ratio : $\dfrac{m \cancel{R} v_t}{q B R^{\cancel{2}}} \sim \dfrac{\rho}{R} \quad \left(\rho \sim \dfrac{m v_t}{q B}\right)$

$p_\phi \sim q A_\phi R = q\, \chi(\psi)$

poloidal flux

flux label

→ confinement to flux surfaces

Definition of quasisymmetry

- basic idea: get conserved canonical momentum for guiding center motion without axisymmetry
- in Boozer coordinates, (ψ, θ, ζ)

geometry enters g.c. Lagrangian $\mathcal{L}$ through $|\vec{B}|$ → it is sufficient to have symmetry of field strength

- definition: $\vec{B}$ is quasisymmetric if

$|\vec{B}|(\psi, \chi)$ in Boozer coordinates (ψ, θ, ζ)

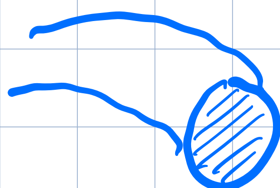
flux label, poloidal, toroidal

$$\chi = M\theta - N\zeta$$

M, N: fixed integers

- define 3rd coordinate, $\eta = \underline{M'}\theta - \underline{N'}\zeta$

s.t. $\dfrac{\partial B(\psi, \chi, \eta)}{\partial \eta} = 0$ → symmetry wrt η

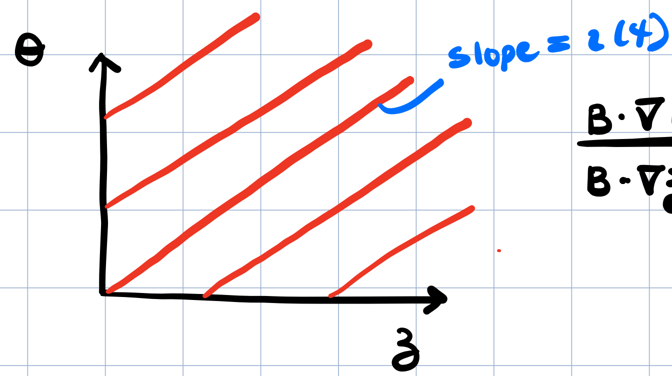


$(M'N - MN' \neq 0$ for well-defined coords.$)$

Quick reminder of Boozer coordinates

- **magnetic coordinates:** choice of flux coordinates (ψ, θ, ζ) s.t. field line dynamics are "straight"

$$\vec{B} = \nabla\psi \times \nabla\theta - \iota(\psi)\nabla\psi \times \nabla\zeta$$



$$\frac{B\cdot\nabla\theta}{B\cdot\nabla\zeta} = \frac{\iota(\psi)\,\cancel{\nabla\psi\times\nabla\theta\cdot\nabla\zeta}}{\cancel{\nabla\psi\times\nabla\theta\cdot\nabla\zeta}} = \iota(\psi)$$

- **Boozer coordinates:** choice of magnetic coordinates s.t. covariant form + Jacobian simplified

$$\vec{B} = K(\psi,\theta,\zeta)\nabla\psi + \underbrace{G(\psi)}_{\text{flux functions}}\nabla\zeta + \underbrace{I(\psi)}_{\text{flux functions}}\nabla\theta$$

$$\sqrt{g} = (\nabla\psi\times\nabla\theta\cdot\nabla\zeta)^{-1} = \frac{G(\psi) + \iota(\psi) I(\psi)}{B^2}$$

→ close connection btw geometry + field strength

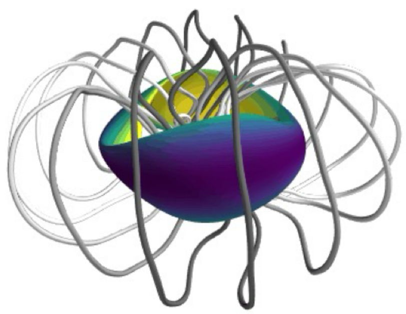
Symmetry helicities

$|\vec{B}|(\psi, \chi)$ $\qquad \chi = M\theta - N\zeta$

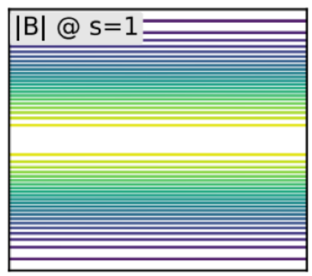
- example: quasiaxisymmetry (QA)

$\chi = \theta \qquad \eta = \zeta \qquad (M=1,\ N=0,\ M'=0,\ N'=-1)$

$$\frac{\partial B(\psi,\theta,\zeta)}{\partial \zeta} = 0$$



θ



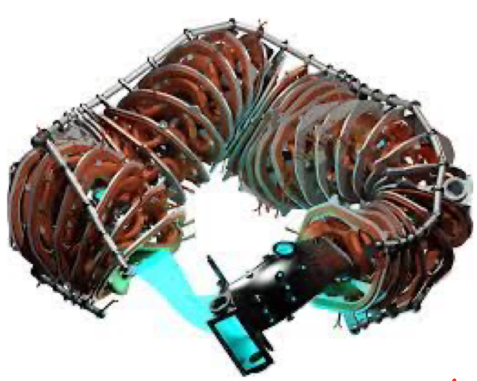
ζ

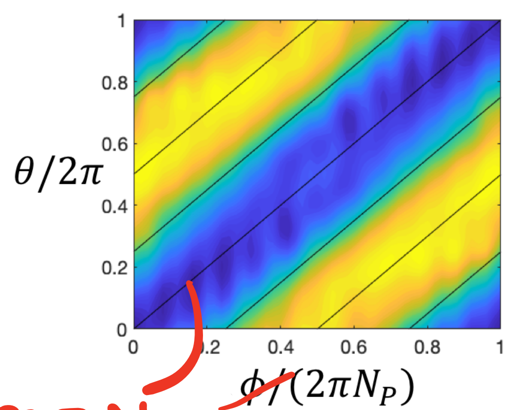
- example: quasi-helical symmetry (QH)

$\chi = M\theta - N\zeta, \quad M=1,\ N \neq 0$

$\eta = \zeta \rightarrow \partial B(\psi,\chi,\zeta)/\partial\zeta = 0$

HSX - U. Wisconsin





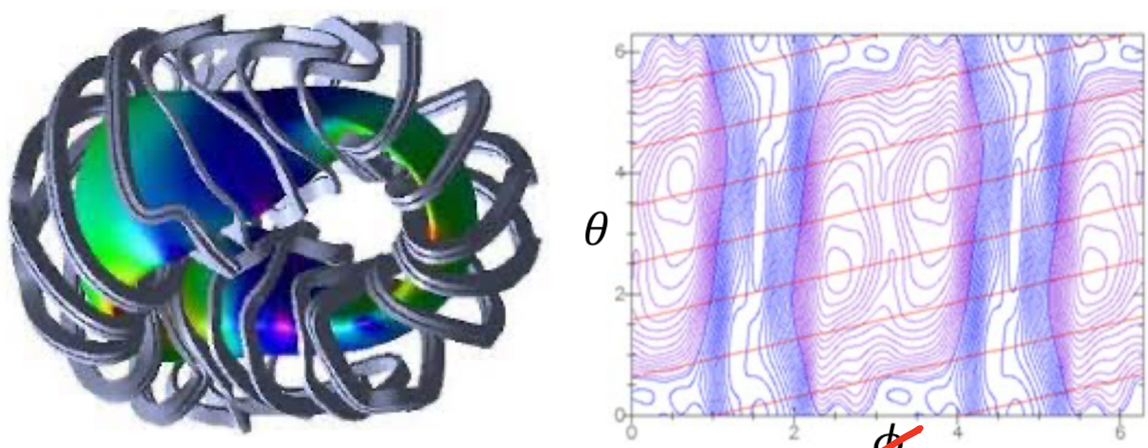
Slope = N

$\cancel{\phi}/(2\pi N_P)$ ζ

- example: quasi-poloidal symmetry (QP)

$$\chi = \zeta \quad \eta = \theta$$

$$\frac{\partial B(\psi, \theta, \zeta)}{\partial \theta} = 0$$



QPS - ORNL design

Guiding center motion in Boozer coordinates

GC Lagrangian

$$\mathcal{L} = (\underset{①}{q\vec{A}(\vec{R})} + \underset{②}{mv_{\parallel}\hat{b}(\vec{R})})\cdot\dot{\vec{R}} - \frac{mv_{\parallel}^2}{2} + \frac{\rho^2 m\dot{\varphi}^2}{2}$$

$$+ \underset{③}{\mu B(\vec{R})}$$

recall : $v_{\parallel}$ = parallel GC velocity
$\vec{R}$ = guiding center position
$\dot{\varphi}$ = gyrofrequency
ρ = gyroradius

- evaluate $\vec{R}$ in Boozer coordinates, providing connection between geometry & $|\vec{B}|$

① $\vec{A}(\vec{R})\cdot\dot{\vec{R}} = \psi\dot{\theta} - \psi_p(\psi)\dot{\zeta}$

② $\hat{b}(\vec{R})\cdot\dot{\vec{R}} = \dfrac{K\dot{\psi} + I\dot{\theta} + G\dot{\zeta}}{B}$

③ $B(\psi, \theta, \zeta)$

- spatial dependence enters $\mathcal{L}$ through:
  ψ, ψ_p, $I(\psi)$, $G(\psi)$ → Flux functions
  $K(\psi, \theta, \zeta)$, $B(\psi, \theta, \zeta)$

- Dependence on (θ, ζ) enters through
  B, K

- MHD force balance ($\vec{J} \times \vec{B} = \nabla p$)
  implies $K = f(\psi, B)$

- if B is QS ($\partial B / \partial \eta = 0$), so is K
  $\left(\frac{\partial K}{\partial \eta} = 0 \right)$

→ QS implies $\frac{\partial \mathcal{L}}{\partial \eta} = 0$

→ E-L : $\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{\eta}} \right] = 0$

- canonical momentum $\partial\mathcal{L}/\partial\dot\eta$ is conserved

$$p_\eta = \partial\mathcal{L}/\partial\dot\eta = q(M\psi_p - N\psi) - \frac{mv_\parallel}{B}(GM + IN)$$

- strong magnetization limit:

$$p_\eta \approx q(M\psi_p - N\psi)$$

Conclusion: symmetry of $|B|$ in Boozer coords, is sufficient for charged particle confinement

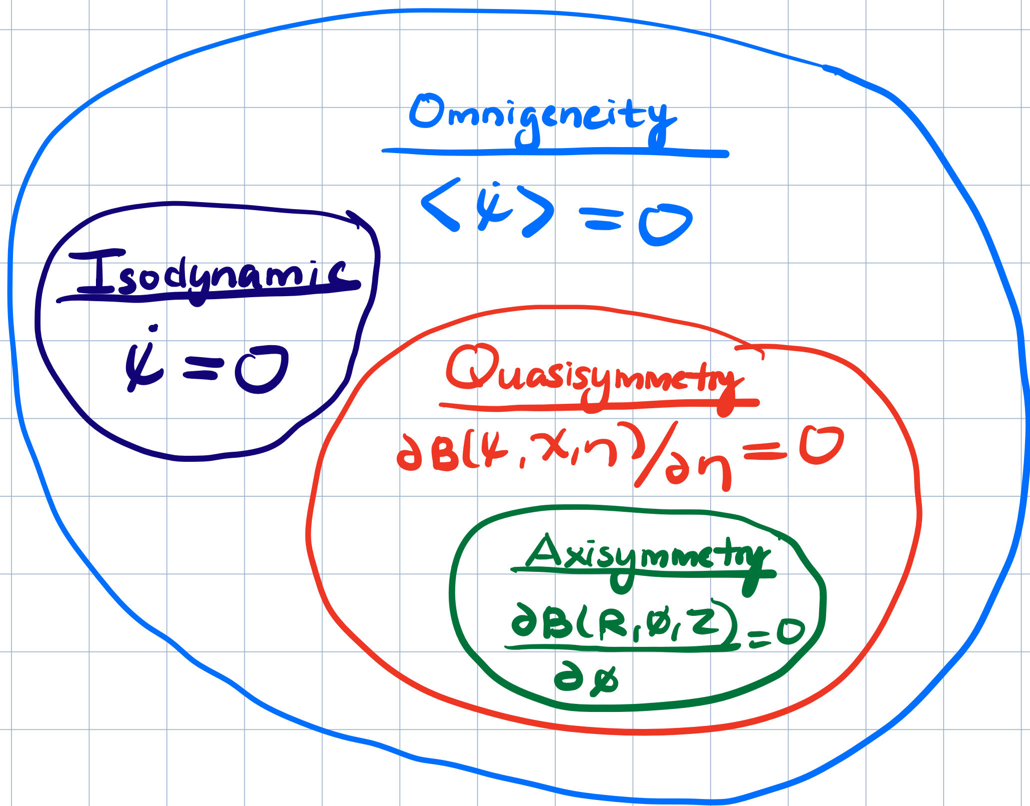
Generalizing QS

- quasisymmetry is sufficient for confinement but not necessary
- necessary condition for averaged confinement:

$$\langle\dot\psi\rangle = \frac{1}{T}\int_0^T dt\,\dot\psi = 0$$

averaged radial drift vanishes

→ Omnigeneity



Bounce-averaged motion

- recall that to lowest order in $\rho/L \ll 1$, charged particles move along field lines
- parallel velocity determined from E & μ conservation

$$E = \frac{m v_{\parallel}^2}{2} + \mu B \Rightarrow v_{\parallel}^2 = \frac{2}{m}(E - \mu B)$$

- mirroring ($v_\parallel = 0$) when $E/\mu = B$

- define $\lambda = \mu/E = 1/B_{crit}$

- at next order in ρ/L, drift across field lines, ∇B & curvature drifts

$$\vec{v}_d \cdot \nabla \psi = \left(v_\parallel^2 + \frac{\mu B}{m} \right) \frac{\vec{B} \times \vec{\nabla} B \cdot \vec{\nabla}\psi}{B^2 \Omega}$$

- to prevent net drift, $\langle \vec{v}_d \cdot \nabla \psi \rangle_t = 0$

- define time average through lowest-order motion along field line

  - recall (ψ, α, ℓ) coordinates:

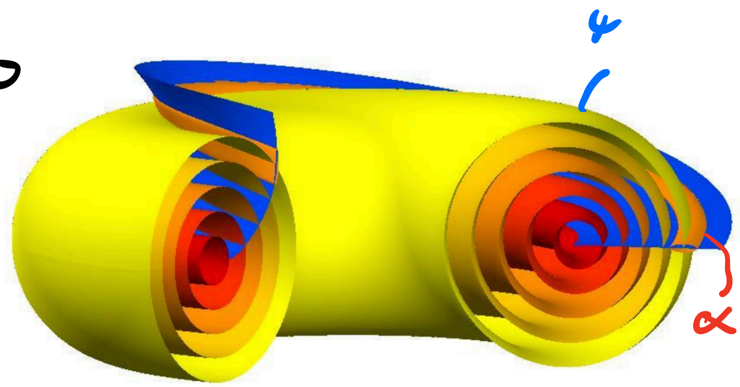
$\vec{B} = \nabla\psi \times \nabla\alpha$

$\rightarrow \vec{B} \cdot \nabla\psi = \vec{B} \cdot \nabla\alpha = 0$

ψ = flux label

α = field line label

ℓ = length along field line



$\rightarrow$ lowest order, along ℓ

for any quantity A,

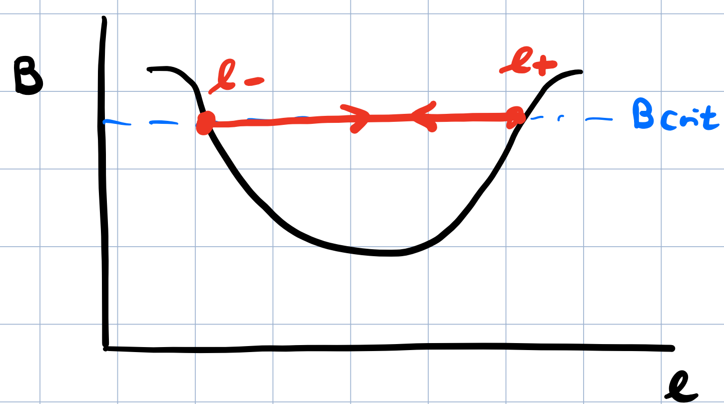
average over periodic loop

$$\langle A \rangle_t = \frac{1}{T}\int_0^T dt\, A = \frac{1}{T}\oint \frac{d\ell}{v_\parallel} A$$

$$T = \oint \frac{d\ell}{v_\parallel}$$ "bounce time"

- $\langle v_d \cdot \nabla \psi \rangle_t$ always vanishes for passing particles on irrational surface

→ focus on trapped particles



when $B = B_{crit}$, $v_\parallel = 0$

$$T = \oint \frac{d\ell}{v_\parallel} = \int_{\ell_-}^{\ell_+} \frac{d\ell}{v_\parallel} + \int_{\ell_+}^{\ell_-} \frac{d\ell}{v_\parallel}$$

- averaged drift in (ψ, α, ℓ) coordinates:

$$\langle \vec{v}_d \cdot \nabla \psi \rangle_t = \frac{m}{qT} \frac{\partial}{\partial \alpha} \left(\oint d\ell \, v_{\parallel} \right)$$

Condition for omnigeneity:

$$\boxed{\frac{\partial}{\partial \alpha} \left(\oint d\ell \, v_{\parallel} \right) = 0 \quad \text{for all } \lambda = B_{crit}^{-1}}$$

Parallel adiabatic invariant

$J_{\parallel} = \oint d\ell \, v_{\parallel}$ is an adiabatic invariant

- adiabatic invariant: conserved quantity associated with approximate periodic motion

- magnetic moment μ is an adiabatic invariant associated with fast gyromotion, $\Omega/\omega \gg 1$
- $J_\parallel$ conserved if $\omega\tau \ll 1$ for characteristic frequencies ω

$$\langle \vec{v}_d \cdot \nabla\psi \rangle_t = -\frac{m}{q\tau} \frac{\partial J_\parallel}{\partial \alpha}$$

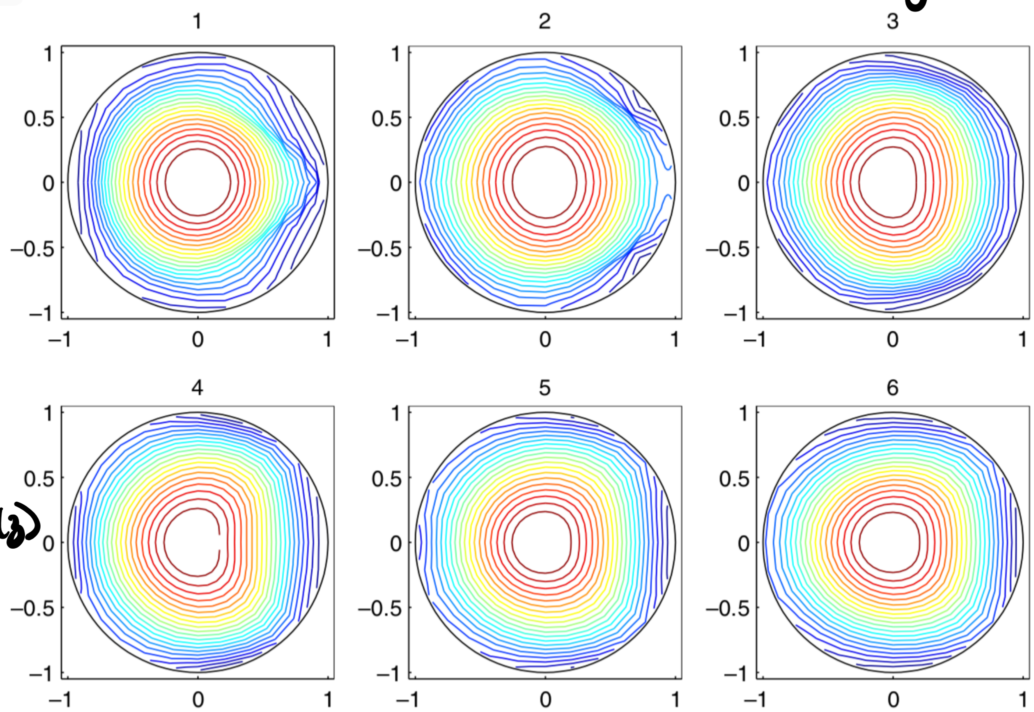
$$\langle \vec{v}_d \cdot \nabla\alpha \rangle_t = \frac{m}{q\tau} \frac{\partial J_\parallel}{\partial \psi}$$

$$\Delta J_\parallel(\psi, \alpha) = \frac{\partial J_\parallel}{\partial \psi} \cancel{\underbrace{\langle v_d \cdot \nabla\psi \rangle}_{\alpha - \frac{\partial J_\parallel}{\partial \alpha}}} + \frac{\partial J_\parallel}{\partial \alpha} \cancel{\underbrace{\langle v_d \cdot \nabla\alpha \rangle}_{\alpha + \frac{\partial J_\parallel}{\partial \psi}}}$$

$\rightarrow J_\parallel$ is invariant

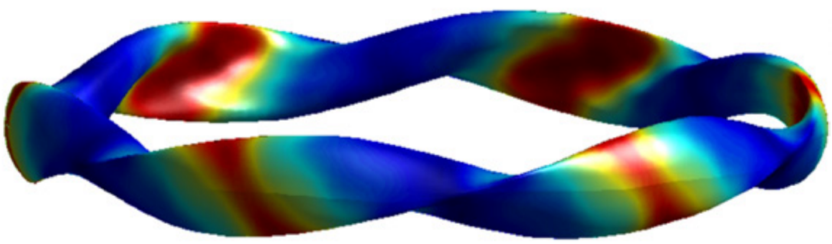
- goal of omnigeneity: align flux surfaces w/ $J_\parallel$ surfaces (drift surfaces)

Drift surfaces in Subbotin QI configuration



$\sqrt{s}\sin(\vartheta)$

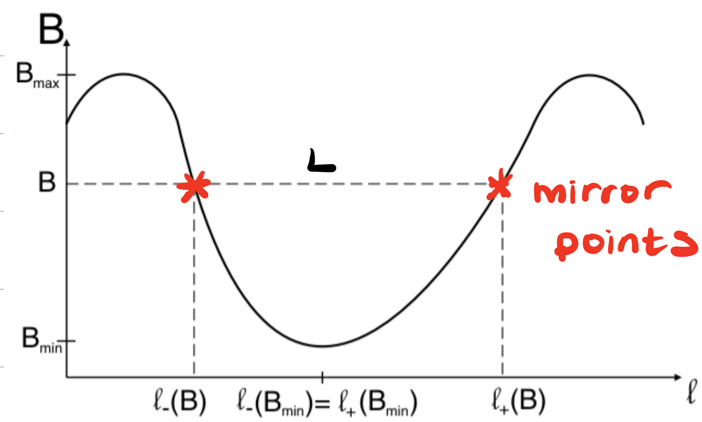
$\sqrt{s}\cos(\vartheta)$ $s = \psi/\psi_b$



quasi-isodynamic (QI) → omnigeneity w/ poloidally-closed contours of $|\vec{B}|$

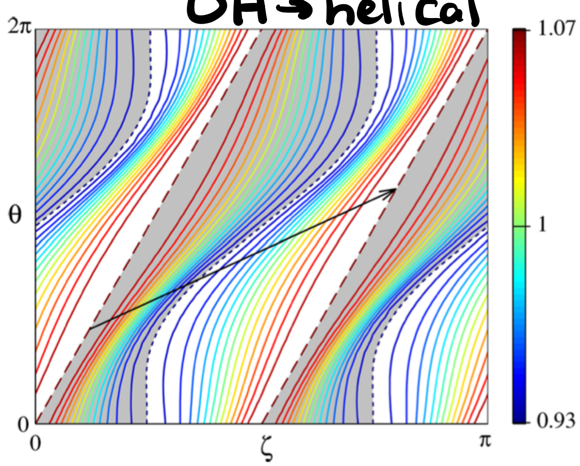
Implications of omnigeneity

- "bounce distance" L is indpt. of field-line label, α

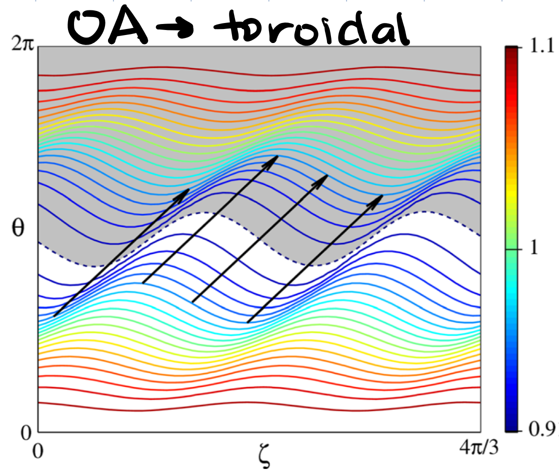


- B_{max} contour is straight in Boozer coordinates
- Other $|B|$ contous must close with same helicity

OH → helical



OA → toroidal



Comparing hidden symmetries

- approximate solutions found numerically for each helicity of QS, omnigeneity
- near the magnetic axis, only QA, QH, + QI possible → other classes more challenging to obtain
- QA: • more compact
  • larger orbit width
  → larger bootstrap
  → worse confinement
- QH/QI: • less compact
  • smaller orbit width